ISRAEL JOURNAL OF MATHEMATICS, Vol. 54, No. 2, 1986

EXTENSIONS OF LIPSCHITZ MAPS INTO BANACH SPACES

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ABSTRACT

It is proved that if $Y \subset X$ are metric spaces with Y having $n \geqq 2$ points then any map f from Y into a Banach space Z can be extended to a map $\hat{f}$ from X into Z so that $\|\hat{f}\|_{\text{lip}} \leqq c \log n \|f\|_{\text{lip}}$ where c is an absolute constant. A related result is obtained for the case where X is assumed to be a finite-dimensional normed space and Y is an arbitrary subset of X.

Consider the following extension problem: given three metric spaces X, Y, Z with $Y \subseteq X$, what is the smallest constant L such that any Lipschitz function $f: Y \to Z$ admits an extension $\tilde{f}: X \to Z$ with $\|\tilde{f}\|_{\text{lip}} \leqq L \|f\|_{\text{lip}}$?

A classical result of Kirszbraum (see [6]) states that if X and Z are Hilbert spaces then $L = 1$. Another classical result is the fact that if $Z = l_\infty^n$ and X, Y arbitrary then $L = 1$ (see [6] again). Recently, this problem gained some new interest. Marcus and Pisier ([4]) proved that for $X = L_p$, $1 < p < 2$, Z a Hilbert space and Y finite (say $|Y| = k$), $L \leqq C(p)(\log k)^{1/p-1/2}$. Johnson and Lindenstrauss ([2]) proved that for X a general metric space, Y and Z as above, $L \leqq C(\log k)^{1/2}$. These two results are close to being the best possible; one has $L \geqq \delta(\log k/\log\log k)^{1/p-1/2}$ in the first setting and $L \geqq \delta(\log k/\log\log k)^{1/2}$ in the second (cf. [2]).

In this note we concern ourselves with a more general situation: Z is a general Banach space and we put no geometrical restrictions on X. We prove the following two theorems:

† Supported in part by US–Israel Binational Science Foundation and by NSF MCS-7903042.
†† Supported in part by NSF MCS-8102714.
Received April 18, 1985

THEOREM 1. *Let $T \subseteq X$ be two metric spaces with $2 \leqq |T| = k < \infty$. Let Z be a Banach space and $f: T \to Z$ a function. Then there exists a function $\tilde{f}: X \to Z$ with $\tilde{f}_{|T} = f$ and*

$$\|\tilde{f}\|_{\mathrm{lip}} \leqq K \cdot (\log k) \cdot \|f\|_{\mathrm{lip}}$$

where K is an absolute constant.

THEOREM 2. *Let X, Z be Banach spaces with $\dim X = n < \infty$, let $F \subseteq X$ be any set and let $f: F \to Z$ be a Lipschitz function. Then there exists a function $\tilde{f}: X \to Z$ with $\tilde{f}_{|F} = f$ and*

$$\|\tilde{f}\|_{\mathrm{lip}} \leqq K \cdot n \cdot \|f\|_{\mathrm{lip}}$$

where K is an absolute constant.

The two theorems have similar proofs. We begin with a proof of Theorem 2. The idea of the proof is based on the proof of an extension theorem of Whitney (see [5], [7]). We begin with a simple covering lemma.

LEMMA 3. *Let X be an n-dimensional normed space with open unit ball B_X. Then there are $\{x_i\}_{i=1}^{\infty}$ in X such that*

(a) $$\bigcup_{i=1}^{\infty} (x_i + B_X) = X$$

and

(b) $$\text{for all } x \in X, \qquad |\{i; x \in x_i + \tfrac{3}{2}B_X\}| \leqq 4^n.$$

PROOF. Let $\{x_i\}_{i=1}^{\infty}$ be a maximal 1-net in X, i.e. $\|x_i - x_j\| \geqq 1$ for $i \neq j$ and for all $x \in X$ there exists an i such that $\|x - x_i\| < 1$. It is then clear that (a) is satisfied.

Let $x \in X$. If $x \in x_i + \frac{3}{2}B_X$ then $x_i \in x + \frac{3}{2}B_X$; it follows that $x_i + \frac{1}{2}B_X \subseteq x + 2B_X$. All the balls $x_i + \frac{1}{2}B_X$ are disjoint so

$$|\{i; x \in x_i + \tfrac{3}{2}B_X\}| \cdot \mathrm{Vol}(\tfrac{1}{2}B_X) \leqq \mathrm{Vol}(2B_X)$$

or

$$|\{i; x \in x_i + \tfrac{3}{2}B_X\}| \cdot 2^{-n} \cdot \mathrm{Vol}(B_X) \leqq 2^n \,\mathrm{Vol}(B_X)$$

and we get the desired result. □

PROPOSITION 4. *Let X be an n-dimensional Banach space and let $F \subseteq X$ be a closed set. Then there exist homothetic copies $\{K_i\}_{i=1}^{\infty}$ of B_X, say $K_i = y_i + \alpha_i B_X$, such that*

(a) $$X \setminus F = \bigcup_{i=1}^{\infty} K_i,$$

(b) $$4 \cdot \alpha_i \leqq d(K_i, F) \leqq 18 \cdot \alpha_i$$

and

(c) $$\text{for all } x \in X \setminus F, \qquad |\{i\,; x \in y_i + \tfrac{3}{2}\alpha_i B_X\}| \leqq 3 \cdot 4^n.$$

PROOF. For $m = 0, \pm 1, \pm 2, \cdots$ let $A_m = \{x\,; C \cdot 2^{-(m+1)} < d(x,F) \leqq C \cdot 2^{-m}\}$ (C will be chosen momentarily). By Lemma 3 there are $x_i^m \in X$ such that for each m, $\{x_i^m + 2^{-m} \cdot B_X\}_{i=1}^{\infty}$ covers X and for all $x \in X$,

$$|\{i\,; x \in x_i^m + \tfrac{3}{2} \cdot 2^{-m} B_X\}| \leqq 4^n.$$

Put

$$K_i^m = x_i^m + 2^{-m} \cdot B_X, \quad \tilde{K}_i^m = x_i^m + \tfrac{3}{2} \cdot 2^{-m} B_X, \quad \textit{and} \quad I_m = \{i\,; K_i^m \cap A_m \neq \emptyset\}.$$

If $i \in I_m$ and $x \in \tilde{K}_i^m$ then

$$(*) \qquad \begin{aligned} (C/2 - 3) \cdot 2^{-m} &\leqq C \cdot 2^{-(m+1)} - \operatorname{diam} \tilde{K}_i^m < d(x,F) \\ &\leqq C \cdot 2^{-m} + \operatorname{diam} \tilde{K}_i^m \leqq (C+3) \cdot 2^{-m}. \end{aligned}$$

For $C \geqq 15$ we have

$$(C+3)2^{-(m+2)} \leqq (C/2 - 3)2^{-m}$$

so that $\tilde{K}_i^m \cap \tilde{K}_j^{m+2} = \emptyset$ for all $i \in I_m$, $j \in I_{m+2}$, $m = 0, \pm 1, \pm 2, \cdots$.

Let $\{K_i\}_{i=1}^{\infty}$ be an enumeration of $\{K_i^m\}_{i \in I_m, m=-\infty}^{\infty}$, then $\{K_i\}_{i=1}^{\infty}$ is the desired covering. Indeed, since for all m

$$A_m \subseteq \bigcup_{i \in I_m} K_i^m \subseteq A_{m-1} \cup A_m \cup A_{m+1},$$

we get that $\bigcup_{i=1}^{\infty} K_i = X \setminus F$ and that for all $x \in X \setminus F$

$$|\{i\,; x \in \tilde{K}_i\}| \leqq 3 \cdot 4^n.$$

Finally, if $K_i = K_i^m$ with $i \in I_m$ then $\alpha_i = \operatorname{rad} K_i = 2^{-m}$ and, by $(*)$ with $C = 15$,

$$4 \cdot \alpha_i = 4 \cdot 2^{-m} \leqq d(K_i, F) \leqq 18 \cdot 2^{-m} = 18 \cdot \alpha_i. \qquad \square$$

Using the covering $\{K_i\}_{i=1}^{\infty}$ we now build a partition of unity for $X \setminus F$. Fix an $m > 1$ to be chosen later and for each i define

$$\varphi_i(x) = \begin{cases} (\tfrac{3}{2}\alpha_i - \|x - y_i\|)^m & \|x - y_i\| \leqq \tfrac{3}{2}\alpha_i \\ 0 & \text{otherwise} \end{cases}$$

and, for $x \in X \setminus F$, define

$$\tilde{\varphi}_i(x) = \varphi_i(x) \Big/ \sum_{j=1}^{\infty} \varphi_j(x)$$

(notice that $\varphi_i(x)$ is non-zero for at most $3 \cdot 4^n$ i's). Then

$$\sum_{i=1}^{\infty} \tilde{\varphi}_i(x) = \chi_{X \setminus F}.$$

LEMMA 5. *For each* $x \in X \setminus F$

$$\limsup_{y \to x} \frac{\sum_{i=1}^{\infty} |\tilde{\varphi}_i(x) - \tilde{\varphi}_i(y)|}{\|x - y\|} \leqq 80m(3 \cdot 4^n)^{1/m} d(x, F)^{-1}.$$

PROOF. First notice that for each i

$$(**) \quad \limsup_{y \to x} \frac{|\varphi_i(x) - \varphi_i(y)|}{\|x - y\|} \leqq \begin{cases} m(\frac{3}{2}\alpha_i - \|x - y_i\|)^{m-1} & \text{if } \|x - y_i\| < \frac{3}{2}\alpha_i, \\ 0 & \text{otherwise.} \end{cases}$$

Now,

$$\sum_{i=1}^{\infty} |\tilde{\varphi}_i(x) - \tilde{\varphi}_i(y)| = \sum_{i=1}^{\infty} \left| \frac{\varphi_i(x)\Sigma\varphi_j(y) - \varphi_i(y)\Sigma\varphi_j(x)}{\Sigma\varphi_j(x)\Sigma\varphi_j(y)} \right|.$$

Add and subtract $\varphi_i(y)\Sigma\varphi_j(y)$ to the numerator of the i-th term to get

$$\sum_{i=1}^{\infty} |\tilde{\varphi}_i(x) - \tilde{\varphi}_i(y)| \leqq \sum_{i=1}^{\infty} \frac{|\varphi_i(x) - \varphi_i(y)|}{\Sigma\varphi_j(x)} + \sum_{i=1}^{\infty} \varphi_i(y) \sum_{j=1}^{\infty} \frac{|\varphi_j(x) - \varphi_j(y)|}{\Sigma\varphi_j(x)\Sigma\varphi_j(y)}$$
$$= 2\sum_{i=1}^{\infty} |\varphi_i(x) - \varphi_i(y)| \Big/ \sum_{j=1}^{\infty} \varphi_j(x).$$

By (**), we get

$$\limsup_{y \to x} \frac{\sum_{i=1}^{\infty} |\tilde{\varphi}_i(x) - \tilde{\varphi}_i(y)|}{\|x - y\|} \leqq \frac{2m\,\Sigma(\frac{3}{2}\alpha_i - \|x - y_i\|)^{m-1}}{\Sigma(\frac{3}{2}\alpha_i - \|x - y_i\|)^m}$$

where the sums in the last expression are taken over all i's such that $\|x - y_i\| < \frac{3}{2}\alpha_i$. Using Hölder's inequality and the fact that for i such that $x \in K_i$, $\frac{3}{2}\alpha_i - \|x - y_i\| \geqq \frac{1}{2}\alpha_i \geqq \frac{1}{40}d(x, F)$ (note $d(x, F) \leqq 2\alpha_i + d(K_i, F) \leqq 20\alpha_i$), we get for $x \in X \setminus F$,

$$\limsup_{y\to x}\frac{\sum_{i=1}^{\infty}|\tilde\varphi_i(x)-\tilde\varphi_i(y)|}{\|x-y\|}\leqq\frac{2\cdot m\cdot(\Sigma(\frac{3}{2}\alpha_i-\|x-y_i\|)^m)^{(m-1)/m}\cdot(3\cdot 4^n)^{1/m}}{\Sigma(\frac{3}{2}\alpha_i-\|x-y_i\|)^m}$$

$$(***)\qquad\leqq\frac{2\cdot m\cdot(3\cdot 4^n)^{1/m}}{(\Sigma(\frac{3}{2}\alpha_i-\|x-y_i\|)^m)^{1/m}}$$

$$\leqq 80\cdot m\cdot(3\cdot 4^n)^{1/m}\cdot d(x,F)^{-1}.$$

□

We are now ready for the

PROOF OF THEOREM 2. There is no loss of generality to assume that F is closed. With the notation as in Proposition 4 and Lemma 5, let $p_i\in F$ be such that $d(p_i,K_i)=d(F,K_i)$. Let $f:F\to Z$ be a Lipschitz map and define $\tilde f:X\to Z$ by

$$\tilde f(x)=\sum_{i=1}^{\infty}f(p_i)\tilde\varphi_i(x)\qquad\text{for } x\notin F$$

(note: the sum is finite) and

$$\tilde f(x)=f(x)\qquad\text{for } x\in F.$$

It is easily checked that $\tilde f$ is continuous. If $x\in X\setminus F$ then $x\in K_{i_0}$ for some i_0 and since $\Sigma\,\tilde\varphi_i(z)=1$ for all $z\in X\setminus F$ we get

$$\limsup_{y\to x}\frac{\|\tilde f(x)-\tilde f(y)\|}{\|x-y\|}=\limsup_{y\to x}\frac{\|\Sigma f(p_i)(\tilde\varphi_i(x)-\tilde\varphi_i(y))\|}{\|x-y\|}$$

$$(****)\qquad=\limsup_{y\to x}\frac{\|\Sigma(f(p_i)-f(p_{i_0}))(\tilde\varphi_i(x)-\tilde\varphi_i(y))\|}{\|x-y\|}$$

$$\leqq\limsup_{y\to x}\,(\sup\|f(p_i)-f(p_{i_0})\|)\cdot\frac{\Sigma|\tilde\varphi_i(x)-\tilde\varphi_i(y)|}{\|x-y\|}$$

where both the sup and the sum extend over all i's such that $\|x-y_i\|\leqq\frac{3}{2}\alpha_i$ or $\|y-y_i\|\leqq\frac{3}{2}\alpha_i$. For i's of the first kind

$$\|y_i-y_{i_0}\|\leqq\tfrac{3}{2}(\alpha_i+\alpha_{i_0})\leqq\tfrac{3}{8}(d(K_i,F)+d(K_{i_0},F))\leqq\tfrac{3}{2}d(x,F)$$

(note: $d(K_i,F)\leqq d(x,F)+2\alpha_i\leqq d(x,F)+\frac{1}{2}d(K_i,F)$ so $d(K_i,F)\leqq 2d(x,F)$). Also

$$\|y_i-p_i\|\leqq d(K_i,F)\leqq 2d(x,F)$$

and similarly

$$\|y_{i_0}-p_{i_0}\|\leqq 2d(x,F).$$

Thus we get

$$\|p_i - p_{i_0}\| \leqq \tfrac{11}{2} d(x, F)$$

and

$$\|f(p_i) - f(p_{i_0})\| \leqq \tfrac{11}{2} d(x, F) \|f\|_{\mathrm{lip}}.$$

If i satisfies $\|y - y_i\| \leqq \frac{3}{2}\alpha_i$ then assuming without loss of generality $y \in K_{i_0}$ and $d(y, F) \leqq 2d(x, F)$ we get similarly

$$\|f(p_i) - f(p_{i_0})\| \leqq 11 \cdot d(x, F) \cdot \|f\|_{\mathrm{lip}}.$$

Returning to (****) and using also (***) we get

$$\limsup_{y \to x} \frac{\tilde{f}(x) - \tilde{f}(y)}{\|x - y\|} \leqq 11 \cdot d(x, F) \cdot \|f\|_{\mathrm{lip}} \cdot 80 \cdot m \cdot (3 \cdot 4^n)^{1/m} \cdot d(x, F)^{-1}$$

$$\leqq 880 \cdot m \cdot (3 \cdot 4^n)^{1/m} \|f\|_{\mathrm{lip}}.$$

Choosing $m = n$ gives the desired result. □

PROOF OF THEOREM 1. Since every metric space embeds isometrically into an L_∞ space, we may assume without loss of generality that X is a Banach space. Order T in some linear order and for $t \in T$ let

$$A_t = \{x \in X \setminus T;\ d(x, t) \leqq d(x, s) \text{ for all } s \in T,\ s \neq t \text{ and } t \text{ is the first to satisfy this}\}$$

and let

$$B_t = \bigcup_{x \in A_t} B(x, \tfrac{1}{2} d(x, T))$$

(where $B(x, r) = \{y \in X; \|y - x\| < r\}$). Then the A_t's are mutually disjoint, $\bigcup_{t \in T} A_t = X \setminus T$ and $\bigcup_{t \in T} B_t = X \setminus T$.

Note that

(*) if $x \in B_t \cap B_s$ then $\|t - s\| \leqq 6 \cdot d(x, T)$.

Indeed, let $y \in A_t$, $z \in A_s$ be such that

$$\|y - x\| < \tfrac{1}{2} d(y, t), \qquad \|z - x\| < \tfrac{1}{2} d(z, s).$$

Then

$$\|y - t\| = d(y, T) \leqq \|y - x\| + d(x, T) \leqq \tfrac{1}{2} d(y, T) + d(x, T)$$

and

$$\|y - t\| = d(y, T) \leqq 2d(x, T).$$

Similarly

$$\| z - s \| = d(z, T) \leqq 2d(x, T)$$

and we conclude

$$\begin{aligned}\| t - s \| &< \| t - y \| + \| y - x \| + \| x - z \| + \| z - s \| \\ &\leqq d(y, T) + \tfrac{1}{2} d(y, T) + \tfrac{1}{2} d(z, T) + d(z, T) \\ &\leqq 6 \cdot d(x, T).\end{aligned}$$

Define now for $t \in T$ and $m > 1$ (to be chosen later)

$$\varphi_t(x) = d(x, B_t^c)^m, \qquad x \in X$$

and for $x \in X \setminus T$

$$\tilde{\varphi}_t(x) = \varphi_t(x) \Big/ \sum_{s \in T} \varphi_s(x).$$

Then

$$\sum_{t \in T} \tilde{\varphi}_{t'} = \chi_{X \setminus T}.$$

As in the proof of Theorem 2 we want first to estimate

$$\limsup_{y \to x} \frac{\Sigma | \tilde{\varphi}_t(x) - \tilde{\varphi}_t(y) |}{\| x - y \|}.$$

A similar computation to that in Theorem 2 gives

$$\limsup_{y \to x} \frac{| \varphi_t(x) - \varphi_t(y) |}{\| x - y \|} \leqq m \cdot d(x, B_t^c)^{m-1}$$

and, for $x \in X \setminus T$,

$$\begin{aligned}\limsup_{y \to x} \frac{\sum_{t \in T} | \tilde{\varphi}_t(x) - \tilde{\varphi}_t(y) |}{\| x - y \|} &\leqq 2 \limsup_{y \to x} \frac{\Sigma | \varphi_t(x) - \varphi_t(y) |}{\| x - y \| \cdot \Sigma \varphi_t(x)} \\ \leqq 2 \cdot m \cdot \frac{\Sigma d(x, B_t^c)^{m-1}}{\Sigma d(x, B_t^c)^m} &\leqq 2 \cdot m \cdot k^{1/m} \cdot \left(\sum_{t \in T} d(x, B_t^c)^m \right)^{-1/m}\end{aligned}$$

(Note: The computation here is actually simpler than the one in the proof of Theorem 2 — one does not have to worry about the number of elements in the summations).

Now, $x \in A_t$ for some t and, by the definition of B_t, $d(x, B_t^c) \geqq \frac{1}{2} d(x, T)$ so that we get

$$(**) \qquad \limsup_{y \to x} \frac{\sum |\tilde{\varphi}_t(x) - \tilde{\varphi}_t(y)|}{\|x - y\|} \leqq 4 \cdot m \cdot k^{1/m} \cdot d(x, T)^{-1}.$$

Define now $\tilde{f}$ by

$$\tilde{f}(x) = \sum_{t \in T} f(t) \tilde{\varphi}_t(x) \qquad \text{for } x \in X \setminus T$$

and

$$\tilde{f}(x) = f(x) \qquad \text{for } x \in T.$$

Then $\tilde{f}$ is continuous and, for $x \in X \setminus T$,

$$\limsup_{y \to x} \frac{\|\tilde{f}(x) - \tilde{f}(y)\|}{\|x - y\|} = \limsup_{y \to x} \Big\| \sum_{t \in T} f(t)(\tilde{\varphi}_t(x) - \tilde{\varphi}_t(y)) \Big\| \Big/ \|x - y\|$$

$$= \limsup_{y \to x} \Big\| \sum_{t \in T} (f(t) - f(s))(\tilde{\varphi}_t(x) - \tilde{\varphi}_t(y)) \Big\| \Big/ \|x - y\|$$

where s is such that $x \in A_s$. The sum above extends over all t such that either $x \in B_t$ or $y \in B_t$. If $x \in B_t$ then since also $x \in B_s$ we have by (*) that $\|t - s\| \leqq 6 \cdot d(x, T)$ and that $\|f(t) - f(s)\| \leqq 6 \cdot d(x, T) \cdot \|f\|_{\text{lip}}$. If $y \in B_t$ and y is close enough to x then $\|t - s\| \cdot 6 \cdot d(y, T) \leqq 7 \cdot d(x, T)$ and $\|f(t) - f(s)\| \leqq 7 \cdot d(x, T) \cdot \|f\|_{\text{lip}}$. From this and (**) we get

$$\limsup_{y \to x} \frac{\|\tilde{f}(x) - \tilde{f}(y)\|}{\|x - y\|} \leqq 7 \cdot d(x, T) \cdot \|f\|_{\text{lip}} \cdot \limsup_{y \to x} \frac{\sum |\tilde{\varphi}_t(x) - \tilde{\varphi}_t(y)|}{\|x - y\|}$$

$$\leqq 7 \cdot d(x, T) \cdot \|f\|_{\text{lip}} \cdot 4 \cdot m \cdot k^{1/m} d(x, T)^{-1}$$

$$\leqq 28 \cdot m \cdot k^{1/m} \cdot \|f\|_{\text{lip}}.$$

Choosing $m = \log k$, we get the desired result. □

We do not know if the results of Theorems 1 and 2 are the best possible, up to absolute constants. Using the fact that the constants in the linear analogues are known, one can show that up to absolute constants, the best constants are no better than $(\log k / \log \log k)^{1/2}$ in Theorem 1 (this is done in [2]) and then $n^{1/2}$ in Theorem 2. (There are couples of Banach spaces $Y \subseteq X$ with $\dim Y = n$ and such that the best *linear* projection from X onto Y has norm $\geqq n^{1/2}$. Now use [3] to show that also the best Lipschitz projection has constant $\geqq n^{1/2}$.) There is a lack of "non-linear" examples in this area. For example, we do not know if in

Theorems 1 or 2 with the additional assumption that X is a Hilbert space one can replace the constants $\log k$ (resp. n) with an absolute constant. We conclude this note with two remarks concerning examples. The first is a non-linear construction, originally presented in [3], which shows that one cannot get a constant better than $(\log k)^{1/4}$ in Theorem 1. (As we remarked above one can actually do better — using a "linear" example.) The proof here is somewhat different from the one in [3].

Consider the metric space T consisting of all subsets of $\{1,\dots,n\}$ with the metric $d(A,B)=h(A,B)^{1/2}$, where h is the Hamming metric $h(A,B)=|A\triangle B|$. Enlarge T by adding abstract elements $\{\tilde{A}\}_{A\subseteq\{1,\dots,n\}}$ satisfying, for some $r>0$ (to be chosen later),

$$(*)\qquad d(A,\tilde{A})\leqq r,\qquad d(\tilde{A},\tilde{B})\leqq\frac{h(A,B)}{r}.$$

Since these two sets of requirements do not contradict the triangle inequality in T (check), we can extend d to get a metric space

$$X=(\{A,\tilde{A}\}_{A\subseteq\{1,\dots,n\}},d)$$

satisfying also $(*)$ (see Corollary 1, p. 271 in [3]). Let $f:T\to l_2^n$ be the map

$$f(A)=\sum_{i\in A}e_i,\qquad A\subseteq\{1,\dots,n\}$$

((e_i) being an orthonormal basis in l_2^n). Then $\|f\|_{\mathrm{lip}}=\|f^{-1}\|_{\mathrm{lip}}=1$. If $\tilde{f}$ is any extension of f to X denote $\lambda=\|\tilde{f}\|_{\mathrm{lip}}$ and $Z_A=\tilde{f}(\tilde{A})$, $A\subseteq\{1,\dots,n\}$. If (A,B) is an edge (i.e., $h(A,B)=1$) then $\|Z_A-Z_B\|\leqq\lambda/r$. If (A,B) is a diagonal (i.e., $h(A,B)=n$) then $\|Z_A-Z_B\|\geqq\sqrt{n}-2\lambda r$. Since in a Hilbert space

$$\left(\underset{(A,B)\text{-diagonal}}{\mathrm{Ave}}\|Z_A-Z_B\|^2\right)^{1/2}\leqq\sqrt{n}\left(\underset{(A,B)\text{-edge}}{\mathrm{Ave}}\|Z_A-Z_B\|^2\right)^{1/2}$$

(see [1]), we get

$$\sqrt{n}-2\lambda r\leqq\sqrt{n}\lambda/r$$

or

$$\lambda\left(\frac{\sqrt{n}}{r}+2r\right)\geqq\sqrt{n}.$$

For $r=n^{1/4}$ we get $\lambda\geqq\frac{1}{3}n^{1/4}$. □

The second remark is that if, in the context of Theorem 1, one takes T to be a discrete set, $1 \geqq d(t,s) \geqq \varepsilon > 0$ for all $t \neq s$, $t, s \in T$. Then one can find an $\tilde{f}$ with $\|\tilde{f}\|_{\mathrm{lip}} \leqq (2/\varepsilon)\|f\|_{\mathrm{lip}}$. This is discussed in [2]. We bring here a very simple proof. Fix a $t_0 \in T$ and define $\tilde{f}$ by

$$\tilde{f}(x) = f(t_0) \qquad \text{if } x \notin \bigcup_{t \in T} B(t, \varepsilon/2)$$

and

$$\tilde{f}(x) = \frac{2}{\varepsilon}\left[d(x,t)\cdot f(t_0) + \left(\frac{\varepsilon}{2} - d(x,t)\right)\cdot f(t)\right] \qquad \text{if } x \in B\left(t, \frac{\varepsilon}{2}\right).$$

It is easily checked that $\|\tilde{f}\|_{\mathrm{lip}} \leqq (2/\varepsilon)\|f\|_{\mathrm{lip}}$. □

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